

Remarks on Duality Transformations and Generalized Stabilizer States

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We consider the transformation of Hamilton operators under various sets of quantum operations acting simultaneously on all adjacent pairs of particles. We find mappings between Hamilton operators analogous to duality transformations as well as exact characterizations of ground states employing non-Hermitian eigenvalue equations and use this to motivate a generalization of the stabilizer formalism to non-Hermitian operators. The resulting class of states is larger than that of standard stabilizer states and allows for example for continuous variation of local entropies rather than the discrete values taken on stabilizer states and the exact description of certain ground states of Hamilton operators.

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I. INTRODUCTION

The investigation of quantum many-body systems with the tools of entanglement theory [1] has recently received considerable attention addressing long-standing questions in the latter employing new methods developed in the former. This includes for example the scaling of block entropies [2] and geometric entropies [3], a new improved understanding and generalization of numerical methods such as DMRG on the basis of matrix-product states [4, 5] and the development of novel approaches based on new classes of efficiently describable quantum states such as weighted graph states [6, 7]. Most of these methods are based on the efficient description of quantum states and are taking place in the Schrödinger picture while the numerical algorithm described in [6] is a mixture of Schrödinger and Heisenberg picture, considering both transformations of Hamiltonians and description of states. In this note consider only transformations on Hamiltonians and operators characterizing states.

II. DUALITY TRANSFORMATIONS

In the following I will consider the effect of sequences of one- and two-qubit quantum gates, such as control phase gates and single qubit operations, in a spirit not dissimilar to the action of a cellular automaton [8] on Hamilton operators. Here and in the following we employ the notation

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

for the Pauli-operators and $\mathbb{1}$ for the identity to simplify notation.

In general, it cannot be expected that with a few simple steps of the nature presented in Fig. 1 one can diagonalize a Hamilton operator (even though later on in this section such an example will be discussed which is then

used to motivate the definition of generalized stabilizer states). It may however be possible to transform Hamilton operators into each another, thus establishing useful equivalences between seemingly different systems. With this aim in mind we would like to explore what effects the application of sequences of quantum gates may have on Hamilton operators describing quantum many body systems. To avoid problems with operator ordering we will consider finite translation invariant systems with open boundary conditions. This will allow me to use settings such as those in Fig. 1 instead of the standard cellular automaton in the Margolus partitioning (see e.g. Fig. 2 and [8] for an excellent exposition of quantum cellular automata and their rigorous definition and classification).

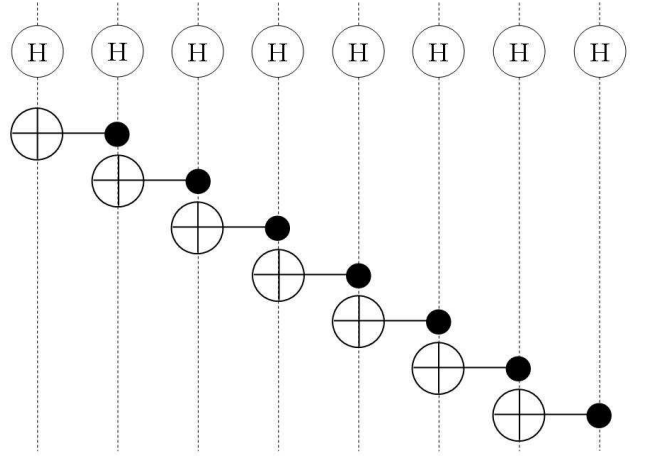


FIG. 1: Time progresses from bottom to the top. Sequence of application of CNOT gates to a set of N qubits followed by the application of Hadamard gates implementing the mapping $H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$.

The effect of the sequence given in Fig 1 is a transfor-

mation

$$\begin{aligned} X_n &\rightarrow \prod_{k=1}^n Z_k \\ Z_n &\rightarrow X_n X_{n+1} \end{aligned}$$

where $Z_{L+1} = \mathbb{1}_L$ if the system consists of L spins and we use the abbreviation $X_k \equiv \mathbb{1}^{\otimes(k-1)} \otimes X \otimes \mathbb{1}^{\otimes(L-k)}$. Connoisseurs will recognize this transformation as a special example of a duality transformation [9] which can map for example the strong coupling regime of a Hamiltonian onto the weak coupling regime. Generalizations of such duality transformations for Ising models with multi-qubit interactions [10] may be constructed easily by similar sequences of CNOT gates. Let us now apply this transformation to the Ising model in transverse magnetic field (whose strength we have set to $B = 1$ for convenience) whose Hamiltonian, with open boundary conditions, is given by

$$H(J) = \sum_{k=1}^{L-1} J X_k X_{k+1} + \sum_{k=1}^L Z_k. \quad (2)$$

Then the above transformation leads to

$$\begin{aligned} H_T(J) &= T H(J) T^\dagger = X_L - J Z_1 + \sum_{k=1}^L J Z_k + \sum_{k=1}^{L-1} X_k X_{k+1} \\ &= X_L - J Z_1 + J H(J^{-1}). \end{aligned}$$

Evidently, the two Hamilton operators are not identical but it is reasonable to expect that in the limit $L \rightarrow \infty$ the two terms X_L and $J Z_1$ may be neglected so that we can state

$$H_T(J) \cong J H(J^{-1}). \quad (3)$$

Because the operators on both sides obey the same algebra this may then be viewed as a statement about the symmetry of the Hamilton operator itself so that we expect for the energy eigenvalues $E(J) = J E(J^{-1})$ [9]. With this relation we may conclude that if the spectral gap of the Hamiltonian vanishes for one set of values J then it will also vanish for J^{-1} [11]. Thus the assumption of a single critical point would then force the conclusion that it must be found at $J = 1$. This turns out to be correct in this case and indeed, near the critical point, the gap above the ground state for example is given by $\Delta E = 2|1 - J^{-1}|$ thus satisfying eq. (3). The validity of these arguments are however perhaps not quite as non-trivial as one may expect as it is not per-se clear that a small perturbation at the boundaries leaves the spectrum essentially unaffected. Such a property requires proof (in fact and perhaps not too surprising this is not correct in the extreme cases $J = 0$ and $J = \infty$) even though one can expect it to hold in most physically reasonable cases.

Now we move beyond self-duality and link the cluster Hamiltonian [12, 13] to the an-isotropic XY model. The cluster Hamiltonian, exhibiting some interesting critical

behaviour [12] and strong finite size effects [13], is defined as

$$H = \sum_{k=2}^{L-1} J X_{k-1} Z_k X_{k+1} + \sum_{k=1}^L B Z_k. \quad (4)$$

Firstly, employing a sequence of controlled phase gates between any neighboring pair reveals that the model is self-dual in the sense of the Ising model discussed above. Employing again the gate sequence in Fig. 1 we find that the Hamiltonian eq. (4) is mapped onto

$$H_T = B X_L - \sum_{k=2}^{L-1} J Y_k Y_{k+1} + \sum_{k=1}^{L-1} B X_k X_{k+1}. \quad (5)$$

Again, in the limit $L \rightarrow \infty$ we expect the correspondence between the cluster Hamiltonian and the anisotropic XY model to become exact and thus relating the critical behaviour of the two models. Note however that the $B X_L$ term cannot be neglected in the extreme limit $J = 0$. Indeed, for $J = 0$ the Hamiltonian in eq. (4) has a unique ground state while eq. (5) without the term $B X_L$ term would be two-fold degenerate.

It is also straightforward to see that by the same transformation the Hamiltonian

$$H = \sum_{k=2}^{L-1} J_1 X_{k-1} Z_k X_{k+1} + \sum_{k=1}^{L-1} J_2 X_k X_{k+1} + \sum_{k=1}^L B Z_k. \quad (6)$$

is mapped to

$$H_T = B X_L - \sum_{k=1}^{L-2} J_1 Y_k Y_{k+1} + \sum_{k=1}^{L-1} B X_k X_{k+1} + \sum_{k=2}^L J_2 Z_k. \quad (7)$$

which is the XY-model in a transverse field whose critical behaviour is well known. Employing operations such as those in figures 1 and 2 one may obtain a large number of relationships between Hamilton operators that become exact in the asymptotic limit. There are various possible directions in which to extend such an approach. One may for example consider the transformations that emerge from the Trotter decomposition of the time-evolution of Hamiltonians H that one has decomposed into two parts H_1 and H_2 such that $H = H_1 + H_2$ and $[H_1, H_2] = 0$ [14]. Strictly, speaking it is not necessary to consider only unitary operations. Needless to say that in this case the spectra of the Hamiltonians are not connected in a very transparent way. The next example demonstrates that such an approach may nevertheless be useful. The following example will also serve to demonstrate that in some cases the above approach actually allows in a simple way to obtain the exact solution of certain Hamilton operators. An example for that, which also serves to motivate the definition of generalized stabilizer states, is

Lemma I – *The ground state $|\Psi\rangle$ of the translation invariant Hamiltonian*

$$H = \sum_{k=1}^N [-J Z_{k-1} X_k Z_{k+1} + B Z_k] \quad (8)$$

with periodic boundary conditions is uniquely determined by the N eigenvalue equations of non-Hermitean operators

$$Z_{k-1} \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}_k Z_{k+1} |\Psi\rangle = |\Psi\rangle \quad (9)$$

for all i and

$$\lambda = -\frac{B}{J} + \frac{J}{|J|} \sqrt{\left(\frac{B}{J}\right)^2 + 1}. \quad (10)$$

Proof: The strategy in the following will be to map the Hamiltonian eq. (8) to a sum of single particle Hamiltonians that can be solved trivially. To this end, consider the operator

$$T = \prod_{j=1}^N \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}_j \prod_{k=1}^N U_{k,k+1} \quad (11)$$

where we define the controlled phase $U_{k,k+1}$ acting on the qubits k and $k+1$ as

$$U_{k,k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (12)$$

From the observation that the well-known cluster-state $|Cluster\rangle = \prod_{k=1}^N U_{k,k+1} |+\rangle^{\otimes N}$ is uniquely determined as the state satisfying for all i the eigenvalue equations $Z_{i-1} X_i Z_{i+1} |Cluster\rangle = |Cluster\rangle$ [15] we immediately find that

$$|\Psi\rangle = T |+\rangle^{\otimes N} \quad (13)$$

satisfies the eigenvalue equations (9). The transformed Hamilton operator $H_T = T^{-1} H T$ is given by

$$H_T = \sum_{k=1}^N \begin{pmatrix} B & -J\lambda^{-1} \\ -J\lambda & -B \end{pmatrix}_k. \quad (14)$$

Now we need to determine λ such that the ground state of H_T is given by $\prod_{j=1}^N |+\rangle_j$. This is easily found to be

$$\lambda = -\frac{B}{J} + \frac{J}{|J|} \sqrt{\left(\frac{B}{J}\right)^2 + 1} \quad (15)$$

and

$$E_0 = -N \sqrt{B^2 + J^2}. \quad (16)$$

The observation

$$H |\Psi\rangle = E_0 |\Psi\rangle \Leftrightarrow H_T |+\rangle^{\otimes N} = E_0 |+\rangle^{\otimes N}. \quad (17)$$

then completes the proof.

Remark I – We could have also applied a different transformation R which applies the local and non-local operations in reverse order to that in T . Indeed

$$\tilde{R} = \prod_{k=1}^N U_{k,k+1} \prod_{j=1}^N U_j \quad (18)$$

where U_j is the unitary transformation that makes

$$U_j^\dagger \begin{pmatrix} B & -J \\ -J & -B \end{pmatrix}_j U_j \quad (19)$$

diagonal and smaller eigenvalue being the vector $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The transformation on the Hamilton operator is presented pictorially in Fig. 2. Thus the state $|\Psi\rangle$ of eq. (9) may also be written as $|\Psi\rangle = R|1\rangle$. It is noteworthy that by interchanging the order in which the single particle and the two-particle operations are applied changes the single particle operator from a non-unitary to unitary. This approach also allows immediately for

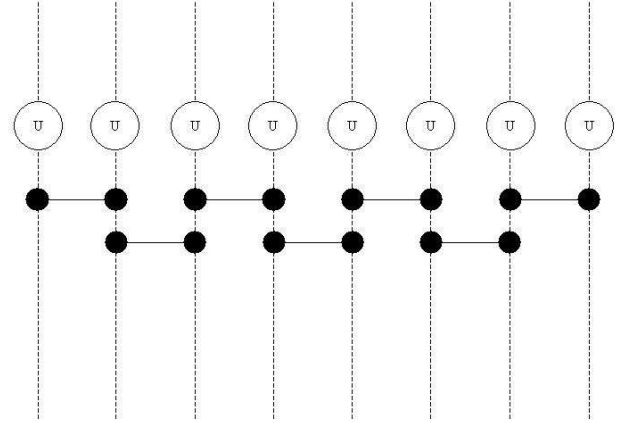


FIG. 2: Time is progressing from bottom to the top. A cellular automaton first applies controlled phase gates (they are symmetric with respect to interchange of control and target) between nearest neighbours and then single qubit gates U such that the combined action maps the Hamilton operator eq. (8) onto a single particle Hamiltonian that is diagonal in the computational basis.

the determination of the entire eigenvalue spectrum. The energy separation between eigenstates is simply given by the splitting between the eigenvalues of $[B - J; -J - B]$. Thus it becomes evident that the Hamilton operator does not possess a quantum critical point.

Remark II – For $J = 0$ the ground state is a product state, while for $B = 0$ the ground state is a cluster state. The transition between those two regimes is continuous and thus we observe that the states $|\Psi\rangle$ can realize local entropies ranging continuously in the interval $[0, 1]$. This is in contrast to standard stabilizer states whose local entropies are quantized in integer units which in itself is enough to see that standard stabilizer states will be ground states to a very limited set of Hamiltonians (see [24] for a much more detailed discussion).

The simple example above suggests that the extension of the stabilizer formalism to non-Hermitean operators may be useful. To remind the reader, a 'stabilizer operator' for an N -particle state is a tensor product of N operators picked from the set $\{X, Y, Z, \mathbb{I}\}$. A set $G = \{g_1, \dots, g_N\}$ of N mutually commuting and independent stabilizer operators [16] is then called a 'generator set' uniquely identifying a state $|\Psi\rangle$ that satisfies $g_k|\Psi\rangle = |\Psi\rangle$ for all $k = 1, \dots, N$. Now we generalize these notions to the definition of *generalized stabilizer states*.

Definition I (Generalized stabilizer states) – For N qubits, a generalized stabilizer state $|\Psi\rangle$ is the unique eigenstate to eigenvalue 1 of the N mutually commuting and independent 'generalized stabilizer operators' $\{g_1, \dots, g_N\}$ where each g_k is an N -fold tensor product of arbitrary, possibly non-Hermitean, linear operators.

From this definition we can immediately draw some straightforward conclusions on some general classes of states that admit such a generalized stabilizer state description.

Remark III – Note that any pure two-qubit state can be written as a generalized stabilizer state. This follows directly from the fact that each pure two qubit state can be obtained from a maximally entangled singlet state by the local application of linear operators. For three qubits all members of the GHZ class are generalized stabilizer states while the members of the W-class may be approximated arbitrarily well by generalized stabilizer states (this follows from the well-known classification of pure three-qubit states under the action of local linear maps [17].) Some interesting results have been obtained recently in [19, 20, 21] concerning the entanglement content of certain stabilizer states and it would be interesting to see how these results may be generalized to the setting of generalized stabilizer states.

Note that one may extend the class of states which admit a description also in another direction, namely by grouping together neighbouring qubits. For kN qubits, a generalized stabilizer state $|\Psi\rangle$ is the unique eigenstate to eigenvalue 1 of the N mutually commuting and independent 'generalized stabilizer operators' $\{g_1, \dots, g_N\}$ where each g_k is an N -fold tensor product of arbitrary, possibly non-Hermitean, linear operators acting on k qubits. It is evident that this will, for $k = 3$ for example, describe all possible three-qubit states but of course this comes at the expense of an exponential increase in complexity of description. In some sense the above concepts of generalized stabilizer states are included in other descriptions of quantum states. In fact, for qubits deformed weighted graph states as they have been introduced in [6] (see also [7, 22] for the concept of weighted graph states) incorporate the generalized stabilizer states according to Definition I. However, in this picture the description is again on the level wave-functions [23], i.e. the Schrödinger picture, while the present approach, via eigenvalue equations, is

situated in the Heisenberg picture. The weighted graph state picture of [6, 7, 22] is probably too general to allow for a detailed quantification of multi-particle entanglement while the generalized stabilizer states may well admit more detailed results for admittedly a less general class of states. This will be the subject of a future publication.

As a final remark it should be noted that by definition, generalized stabilizer states possess a unique characterization of the quantum state of an n -qubit system employing only resources that are polynomial in n . This alone, however, is not sufficient for applications. It is also important to be able to derive relevant physical quantities directly from the stabilizer formalism. Indeed, having first to deduce the state explicitly and then computing the property from the state would generally involve an undesirable exponential overhead in resources. While one can expect a direct approach to be possible in principle, it is evident that detailed and explicit presentations of algorithms to achieve these tasks in a systematic way and whose convergence is proven are of interest. For standard stabilizer states such a program has been carried out and detailed algorithms have been provided [18]. There, normal forms in the context of bi-partite entanglement have been found together with algorithms with proven convergence to obtain these. These tools may be transferred directly to generalized stabilizer states whenever they are obtained from standard stabilizer states by the local application of linear operators.

III. CONCLUSIONS

In this work we have studied duality relations between Hamilton operators from the viewpoint of sequences of quantum operations. The close relationship of these transformations to duality transformations in Hamilton operators has been noted. The same approach has then been used to derive an exact solution for a many-body Hamiltonian which has led to the characterization of the ground state via eigenvalue equations using non-Hermitean operators. This motivated the definition of a generalization of the concept of stabilizer states to incorporate non-Hermitean operators. The properties of such states and the potential offered by this approach still remain to be explored.

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